Caged black hole thermodynamics: charge, the extremal limit, and finite size effects

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ABSTRACT: We extend the effective field theory treatment of the thermodynamics of small compactified black holes to the case of charged black holes. The relevant thermodynamic quantities are computed to second order in the parameter $\lambda \sim (r_0/L)^{d-3}$. We discuss how the addition of charge to a caged black hole may delay the phase transition to a black string. In the extremal limit, we construct an exact black hole solution which serves as a check for our perturbative results. Finite size effects are also included through higher order operators in the worldline action. We calculate how the thermodynamic quantities are modified in the presence of these operators, and show they enter beyond order λ^2 as in the uncharged case. Finally, we use the exact solution to constrain the Wilson coefficients of the finite size operators in the extremal limit.

Contents

1 Introduction

The physics of black objects in higher dimensional spacetimes can be much richer than in four dimensions. In higher dimensions, it is possible to find black objects of different horizon topologies and transitions between these different phases. One extensively studied example can be seen in the rotating stationary solutions of [1], where both black hole (BH) and black ring solutions exist. For particular combinations of mass and angular momentum both phases can exist simultaneously and this has lead to a rich phase diagram.

Another important system with black objects of different topologies and interesting transitions is the BH-black string (BS) system; see [2] and references therein. Here a spacetime with a single compact dimension is considered, namely $\mathbb{R}^{d-2,1} \times \mathbb{S}^1$, where d is the total spacetime dimension. Beyond $\mathbb{R}^{d-1,1}$, this is the simplest possible extension to the spacetime manifold, and thus presents a natural candidate for study. In this paper we also calculate in this spacetime. Note that in the case of a more elaborate manifold compactification, one expects the transition physics to be similar to the $\mathbb{S}^1$ case.

In the BH-BS system, the BH and BS each possess distinct topologies – the BH horizon has a spherical topology $\mathbb{S}^{d-2}$ and the BS has a horizon topology $\mathbb{S}^{d-3} \times \mathbb{S}^1$. The stability

of both phases depends on the ratio of the two scales in the problem, the mass M and the size of the compact dimension L. For large masses $GM \gg L^{d-3}$, a uniform BS is the only stable solution. The analytical solution in this region of phase space is well-known and exhibits a Gregory-Laflamme instability at lower masses [3]. For small masses $GM \ll L^{d-3}$ however, the only stable phase is a BH. Since this BH is localized in a compact dimension, it is referred to as a caged BH or Kaluza-Klein (KK) BH. Moreover, in an intermediate mass region, there is an extra phase corresponding to a non-uniform BS, where the non-uniformity exists in the compact dimension. This phase may be stable or unstable depending on the total number of spacetime dimensions.

Recently, the phase transitions between the uniform BS, the non-uniform BS, and the caged BH have been discussed extensively. In particular, numerical research of the BH/BS system has shown the BH and the non-uniform BS phases merge at a topology changing transition [4]. Although no closed form metric exists for a caged BH, it is possible to calculate the properties of *small caged BHs* in a perturbation expansion in the parameter $\lambda \sim (r_0/L)^{d-3}$. Here r_0 is the Schwarzschild radius and L is the size of the compact dimension. Note that in the region where the BH/BS phase transition occurs, perturbative methods break down and numerical methods or other non-perturbative analytical approaches [5] must be employed.

There exist several methods to compute the perturbative expansion. In [6], this problem was studied within adapted coordinates in a single patch. In [7], the thermodynamics were computed by applying the matched asymptotic expansion method. Finally [8] used an effective field theory (EFT) treatment to calculate the thermodynamics of neutral caged BHs. This last approach appears to be the most systematic and transparent and we employ it here. It applies the worldline EFT methods used to describe extended objects in general relativity (GR), originally proposed by Goldberger and Rothstein [9, 10] (see [11] for a pedagogical introduction).

In [8], the EFT method was used to study the thermodynamics of static caged BHs without charge up to $\mathcal{O}(\lambda^2)$ in the perturbation expansion. The method was further optimized in [12], where a metric parametrization using a temporal KK reduction was introduced. This parametrization improves EFT calculations in static or weakly time dependent systems [13]. The leading contributions to the thermodynamics of small rotating caged BHs were also studied in [12].

In addition to the thermodynamic calculations mentioned above, the worldline EFT approach to GR has been successfully applied to the post-Newtonian expansion of binary systems [9, 13]. Spin degrees of freedom have also been incorporated into the EFT framework [14, 15] and gravitational radiation has been studied [9, 16]. The self force on a compact object was also computed [17] and absorptive effects were introduced in [18, 19]. Finally, the energy momentum tensor was considered in [20] and recently, [21] showed how to constrain the three and four graviton vertices.

In this paper, we extend the perturbative studies of [8] and [12] to electrically U(1) charged caged BHs. Little is known about the phase diagram of this system, but a richer phase structure can be expected, due to the presence of the charge as an additional parameter. The stability of the uniform charged BS has been considered [22] and similar properties

to the uncharged case were found. Recently, [23] used a Harrison transformation to study charged black objects on KK spacetimes in Einstein-Maxwell-dilaton gravity.

This work contains three main parts. First we compute all thermodynamic quantities to $\mathcal{O}(\lambda^2)$ in the presence of charge. Our results are valid for any value of the charge between zero and its extremal value as long as λ is small. Note that to $\mathcal{O}(\lambda^2)$, the corrections stem purely from the point particle description of the BH in the EFT framework. For vanishing charge we recover the results of [8, 12].

Next, we construct an exact charged caged BH solution in the extremal limit and compute its thermodynamics. In this limit, the mass, tension, and electrostatic potential do not get renormalized, i.e. they are simply given by their expressions in uncompactified space. These results provide a further check of our perturbative calculations.

Beyond $\mathcal{O}(\lambda^2)$, finite size effects are expected to become important and we dedicate the last section of our paper to them. In the EFT approach, finite size effects are simply represented by higher order terms in the worldline action. We evaluate the contributions of the leading order finite size operators to various thermodynamic quantities for both uncharged and charged caged BHs. The final results are expressed in terms of undetermined matching coefficients, which need to be computed via a matching procedure. Of particular note is the uncharged caged BH, where there is a non-renormalization of the entropy at leading order in the finite size operators. Finally, for a small charged caged BH in the extremal limit, we undertake a matching calculation of the finite size coefficients using our exact solution. Due to a degeneracy in the thermodynamic relations, we are unable to uniquely fix all three coefficients in the extremal limit.

2 Effective field theory setup

In this section we will establish our conventions and the main ingredients for our calculation. The general prescription of how to set up an EFT for small KK BHs is explained in [8] and [12] and we refer the reader to these papers for the details such as the power counting.

We restrict our attention to the simplest compactification in higher dimensional spacetime, that of $\mathbb{R}^{d-2,1} \times \mathbb{S}^1$. In this spacetime a hierarchy of scales forms when we consider a small BH with Schwarzschild radius $r_0 \ll L$, where L is the asymptotic size of the compactified dimension. With such a hierarchy in place, caged BHs can be treated within an EFT that allows their properties to be calculated perturbatively in the expansion parameter λ, which is defined by

$$\lambda := \left(\frac{r_0}{L}\right)^{d-3} \zeta(d-3), \tag{2.1}$$

with $\zeta(z)$ denoting the Riemann zeta function. The Schwarzschild radius r_0 is given by

$$r_0^{d-3} = \frac{16\pi G\, m_0}{(d-2)\Omega_{d-2}}, \quad \Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}, \tag{2.2}$$

where m_0 is the mass of the BH in an uncompactified d-dimensional spacetime. In the language of quantum field theory m_0 is called the bare mass of the BH. The charge of the BH is Q. Coordinates on the d-dimensional spacetime are $x^\mu = (x^0, \mathbf{x})$ where $\mathbf{x} = (\mathbf{x}_\perp, z)$,

with z labeling the coordinate along $\mathbb{S}^1$. We let Greek indices run over all coordinates while Latin indices denote spatial components.

The setup of the EFT now proceeds as in [8, 12] but we must also include the U(1) gauge interaction. Instead of obtaining a BH solution by solving the vacuum Einstein-Maxwell equations, we integrate out the BH, i.e. we integrate out the short distance scale r_0. This is achieved by replacing the BH by an effective theory consisting of the worldline of a massive charged particle coupled to the Einstein-Maxwell system. The action is then given by $S = S_{\text{EH-EM}} + S_{\text{BH}}$ with

$$S_{\text{EH-EM}}[g_{\mu\nu}, a_\mu] = \frac{-1}{16\pi G} \int d^d x \sqrt{|g|} \left(R[g] + f^{\mu\nu} f_{\mu\nu} \right), \tag{2.3}$$

$$S_{\text{BH}}[x, g_{\mu\nu}, a_\mu] = -m_0 \int d\tau + Q \int dx^\mu a_\mu + \dots, \tag{2.4}$$

where $f_{\mu\nu}$ is the Maxwell field strength tensor, a_μ is the electromagnetic vector potential, and $d\tau = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$. While the first two terms in S_{BH} describe a charged massive point particle, the ellipsis denote higher order terms which encode finite size effects. These terms are discussed in section 5.

Following [12] we exploit time translation symmetry and apply the standard KK ansatz for the metric

$$ds^2 = e^{2\phi}(dt - A_i dx^i)^2 - e^{-2\phi/(d-3)} \gamma_{ij} dx^i dx^j, \tag{2.5}$$

reducing over the non-compact temporal coordinate. We can write $\gamma_{ij} = \delta_{ij} + \sigma_{ij}$ so that flat Minkowski spacetime is given by the ground state $\phi = A_i = \sigma_{ij} = 0$. This parametrization reorganizes the dynamical components of the metric, i.e. $g_{\mu\nu} \rightarrow (\phi, A_i, \sigma_{ij})$, into a form which is advantageous from a computational perspective [12, 13]. In addition, we also decompose the electromagnetic d-vector potential into temporal and spatial components $a_\mu = (\varphi, a_i)$. Note the distinction between the notation for the KK gravitational scalar field ϕ and the electromagnetic scalar field φ.

It is convenient to parameterize the worldline using coordinate time $t = x^0$. Since our problem is static, we use the d-velocity $dx^\mu/dt = (1, 0)$ and we may fix the location of the BH at $\mathbf{x} = 0$ without loss of generality. As a consequence, the fields A_i, a_i, and σ_{ij} cannot couple to the worldline as long as we neglect finite size effects, and we will drop the dependence on x in S_{BH}. Staticity also implies there is the additional symmetry of time reversal symmetry $\mathbf{T}$, which means both vector fields A_i and a_i, which are odd under $\mathbf{T}$, can be set to zero in any purely classical calculation like the one here. In the static limit the action is proportional to $\int dt$ and we will suppress this component from this point. The resulting action takes the form

$$S_{\text{EH-EM}}[\phi, \sigma_{ij}, \varphi] = \frac{-1}{16\pi G} \int d^{d-1} \mathbf{x} \sqrt{|\gamma|} \left(-R[\gamma] + \frac{d-2}{d-3} (\partial\phi)^2 - 2e^{-2\phi} (\partial\varphi)^2 \right), \tag{2.6}$$

$$S_{\text{BH}}[\phi, \varphi] = -m_0 e^\phi + Q\varphi + \dots, \tag{2.7}$$

where $(\partial\phi)^2 = (\partial_i\phi)(\partial_j\phi)\gamma^{ij}$ and similarly for $(\partial\varphi)^2$. We do not have to gauge fix the action because the only propagators used in our calculations are the ϕ and φ propagators, and these can be easily computed from (2.6).
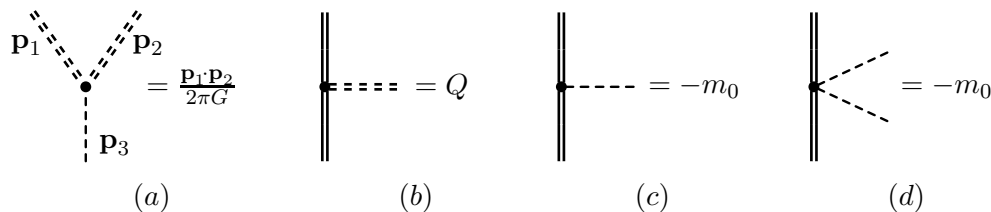
Figure 1. Feynman rules needed for the charged caged BH thermodynamics computations to $\mathcal{O}(\lambda^2)$. The single dashed lines correspond to the KK scalar field ϕ, which is the propagating scalar mode in the gravitational sector and the double dashed lines denote the electromagnetic scalar φ. The Feynman rule (a) is given in momentum space, where we suppress the delta function, and (b)–(d) are in coordinate space.

The action we introduced above describes the dynamics at length scales larger than r_0. In order to compute an observable measured at infinity, we now have to integrate out the scale L. This is performed by splitting up the fields into short and long wavelength modes, e.g. we write $\phi = \bar{\phi} + \phi_L$ where $\bar{\phi}$ is the long wavelength mode and ϕ_L is the short wavelength mode which we will integrate out. We will drop the index L from now on. The formal expression for the effective action is given by

$$e^{iS_{\text{eff}}[\bar{\phi},\bar{\sigma}_{ij},\bar{\varphi}]} = \int \mathcal{D}\phi \mathcal{D}\sigma_{ij} \mathcal{D}\varphi e^{iS[\bar{\phi}+\phi,\bar{\sigma}_{ij}+\sigma_{ij},\bar{\varphi}+\varphi]}, \qquad (2.8)$$

and clearly S_{eff} will contain the standard Einstein-Maxwell action $S_{\text{EH-EM}}[\bar{\phi},\bar{\sigma}_{ij},\bar{\varphi}]$. The BH dynamics is contained in $S_{\text{eff,BH}}$, which in practice is expanded in powers of the long wavelength modes and the terms in this expansion are computed via Feynman diagrams. These terms are then related to the required observable. For example, the mass is simply given by the constant term in $S_{\text{eff,BH}}$ without any couplings to the long wavelength modes.

In [12] it was shown the KK parametrization of the metric in (2.5) reduces the number of diagrams to be calculated in the uncharged case. This remains true when we consider charged compactified BHs, however there will be diagrams at $\mathcal{O}(\lambda^2)$ which are not of an explicitly factorized form. These diagrams arise due to a vertex with two electromagnetic scalar fields φ and one gravitational field ϕ derived from (2.6).

We now proceed to write down the Feynman rules derived from the action (2.6) and (2.7). The long wavelength modes are not needed until section 5 so we set them to zero for now. The procedure for deriving the Feynman rules is discussed in [8, 12] and we simply summarize the results in figure 1. Note we follow the conventions of [12] when deriving the Feynman rules. In contrast to the gravitational couplings, it is only possible to have one photon coupled to the worldline at a given interaction point and we also have a $\phi\varphi\varphi$ vertex as mentioned earlier. The propagator in position space for the gravitational ϕ field is

$$D_\phi(\mathbf{x}_\perp - \mathbf{x}'_\perp; z - z') = \frac{8\pi G}{L} \frac{d-3}{d-2} \times \qquad (2.9)$$
$$\times \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{k}_\perp}{(2\pi)^{d-2}} \frac{1}{\mathbf{k}_\perp^2 + (2\pi n/L)^2} e^{i\mathbf{k}_\perp \cdot (\mathbf{x}-\mathbf{x}')_\perp + 2\pi i n(z-z')/L},$$

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and for the electromagnetic scalar φ the propagator is

$$D_\varphi(\mathbf{x}_\perp - \mathbf{x}'_\perp; z - z') = -\frac{4\pi G}{L} \times \tag{2.10}$$
$$\times \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{k}_\perp}{(2\pi)^{d-2}} \frac{1}{\mathbf{k}_\perp^2 + (2\pi n/L)^2} e^{i\mathbf{k}_\perp \cdot (\mathbf{x}-\mathbf{x}')_\perp + 2\pi in(z-z')/L}.$$

In the Feynman diagrams, single dashed lines correspond to the gravitational scalar field ϕ, whereas double dashed lines denote the electromagnetic scalar φ. The worldlines, which are denoted by solid double lines, do not propagate so there are no propagators associated with these lines.

3   Thermodynamics of charged caged BHs from EFT

To compute the thermodynamics of small charged caged BHs we first calculate diagrammatically its ADM mass and the redshift factor. The mass, for instance, is computed by integrating out short wavelength modes of order L, as described in (2.8). Once these quantities are at hand, one can calculate the remaining thermodynamics.

3.1   The ADM mass at $\mathcal{O}(\lambda^2)$

The ADM mass m can be read off from the constant term in the effective action $S_{\text{eff,BH}}$ [12] and to $\mathcal{O}(\lambda^2)$ this equals the sum of classical vacuum diagrams shown in figure 2. In the uncharged case only diagrams in figure 2(a) and (b) contribute and these diagrams are calculated to be [12]

$$\text{Figure } 2(a) = +\frac{m_0}{2}\lambda, \tag{3.1}$$

$$\text{Figure } 2(b) = -\frac{m_0}{2}\lambda^2, \tag{3.2}$$

where the parameter λ is given by (2.1) and we have used the Feynman rules of section 2, along with (A.1).

For the charged BH there are two additional diagrams to compute, figure 2(c) and (d). The computation of diagram figure 2(c) is similar to figure 2(a) and the result is

$$\text{Figure } 2(c) = -\frac{Q^2}{4m_0}\frac{d-2}{d-3}\lambda. \tag{3.3}$$

It can be understood in terms of the total electrostatic energy $E = \frac{1}{2}\sum_i Q\,\varphi_i(O)$, where the sum runs over all images of the BH in the covering space and $\varphi_i(O)$ represents the electrostatic potential created by the image i at the location of the BH.

Diagram figure 2(d) is slightly more complicated, so we calculate it here explicitly. The symmetry factor we divide by is 2. We can now write the amplitude

$$-(4\pi GQ)^2 \frac{d-3}{d-2}\, m_0 \sum_{\mathbf{p}_1, \mathbf{p}_2} \frac{2\,\mathbf{p}_1 \cdot \mathbf{p}_2}{\mathbf{p}_1^2\,\mathbf{p}_2^2(\mathbf{p}_1+\mathbf{p}_2)^2}, \tag{3.4}$$
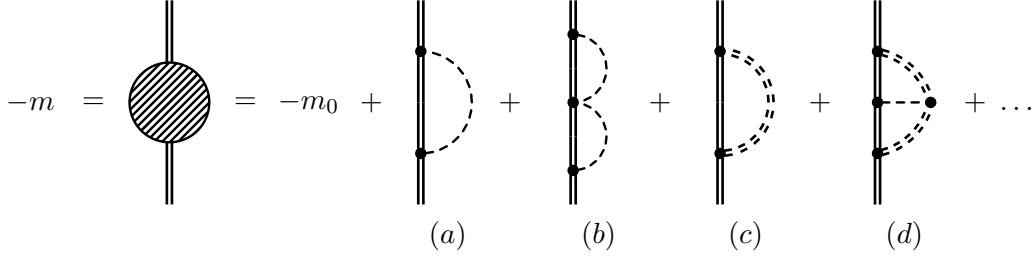
Figure 2. Diagrams that contribute to the ADM mass to $\mathcal{O}(\lambda^2)$.

where $\mathbf{p} = \left(\mathbf{p}_\perp, \frac{2\pi n}{L}\right)$ and

$$\sum_{\mathbf{p}} \!\!\!\!\!\!\int = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}}.$$

By using the identity $2\,\mathbf{p}_1 \cdot \mathbf{p}_2 = (\mathbf{p}_1 + \mathbf{p}_2)^2 - \mathbf{p}_1^2 - \mathbf{p}_2^2$ and then subsequently redefining the momentum, we find this amplitude can be written in a factorized form

$$+ (8\pi G Q)^2 \frac{d-3}{d-2} m_0 \left[\frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}} \frac{1}{\mathbf{p}_\perp^2 + (2\pi n/L)} \right]^2, \tag{3.5}$$

and with the identity (A.1) it leads to

$$\text{Figure } 2(d) = + \frac{Q^2}{4m_0} \frac{d-2}{d-3} \lambda^2. \tag{3.6}$$

Combining these results gives the following expression for the ADM mass to second order in λ

$$m = m_0 \left[1 - \frac{1}{2} \eta^2 \lambda + \frac{1}{2} \eta^2 \lambda^2 + \ldots \right]. \tag{3.7}$$

The variable η in the above expression plays the role of the order parameter and is given by

$$\eta = \sqrt{1 - \frac{1}{2} \frac{d-2}{d-3} \frac{Q^2}{m_0^2}}. \tag{3.8}$$

It vanishes in the extremal limit when

$$\frac{1}{2} \frac{d-2}{d-3} Q^2 = m_0^2 \tag{3.9}$$

holds and reaches its maximum value, $\eta = 1$, for an uncharged BH. Note also, corrections to the mass of an extremally charged BH vanish to second order identically, namely the ADM mass is left unrenormalized to $\mathcal{O}(\lambda^2)$ in the extremal limit. As we show in section 4, it turns out such a non-renormalization of the ADM mass must hold to any order in the perturbation expansion.
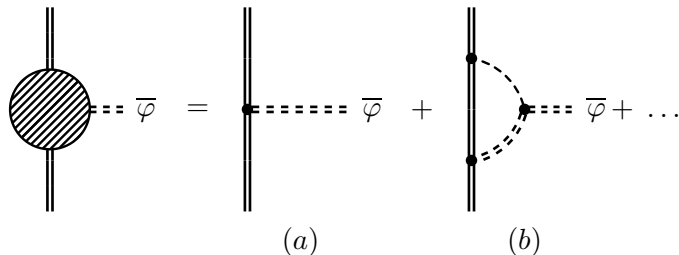
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Figure 3. Diagrams that contribute to the asymptotic value of the electric charge.

3.2 Thermodynamics

In the previous subsection we computed the BH mass (3.7) in terms of the charge Q, the compactification parameter L, and the bare mass m_0, however this relation does not fully determine the thermodynamics of the system. In this section we will compute the remaining thermodynamic quantities, the temperature T, the entropy S, the electrostatic potential Φ, and the tension $\hat{\tau}$. We also give the expression for the Gibbs potential G.

First, we will clarify the notion of the charge Q. In our calculations there is no distinction between the bare and asymptotic values of the electric charge Q. Intuitively such a non-renormalization stems from the electric flux conservation requirement. Technically, the asymptotic value of the electric charge can be read off the following linear term in the effective action

$$S_{\text{eff,BH}}[\bar{\phi}, \bar{\sigma}_{ij}, \bar{\varphi}] \supset +Q\bar{\varphi}. \tag{3.10}$$

This term is the sum of Feynman diagrams like those of figure 3 with $\bar{\varphi}$ set to a constant, in which case the momentum flowing into the diagram vanishes. All couplings involving gravitons and photons are derived from the second term in (2.3). Since all of these vertices contain derivatives acting on the electromagnetic scalar field $\bar{\varphi}$, they all vanish in the zero momentum limit. The only possible vertex for a photon coupling to the worldline is figure 1(b) which cannot include any additional gravitons. Thus only the leading order diagram is nonzero and the sub-leading diagrams such as figure 3(b) vanish. This means the asymptotic value of the electric charge is equal to its bare value Q. The inclusion of finite size effects does not alter this conclusion, since all higher order operators with electromagnetic interactions include derivatives acting on $\bar{\varphi}$.

Next we compute the asymptotic temperature. It is canonically conjugate to time and hence transforms inversely to t, namely $T/T_0 = t_0/t$. Therefore one can relate the asymptotic temperature T to the local temperature T_0 using the redshift,

$$T = R\,T_0. \tag{3.11}$$

As long as we do not include finite size effects, the horizon will not be deformed and the local temperature will coincide with the temperature of an uncompactified d-dimensional Reissner-Nordström BH [24],

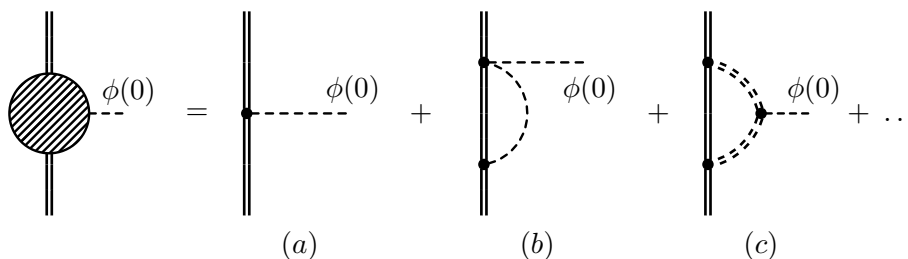$$T_0(m_0, Q) = T(L \to \infty) = \frac{d-3}{d-2}\frac{m_0}{S_0}\eta, \tag{3.12}$$

Figure 4. Diagrams required to compute the redshift factor R to $\mathcal{O}(\lambda^2)$. Here, $\phi(0)$ denotes that the ϕ field propagates back to $\mathbf{x} = 0$ on the BH worldline but does not have a vertex.

where η is the order parameter defined in (3.8) and S_0 is the entropy of an uncompactified d-dimensional charged BH given by

$$S_0(m_0, Q) = \frac{k_B}{4G} \Omega_{d-2} \, r_+^{d-2}, \qquad r_+^{d-3} = \frac{8\pi G m_0}{(d-2)\Omega_{d-2}} (1 + \eta). \tag{3.13}$$

Note the local temperature T_0 in (3.12) and the expression for S_0 in (3.13) are correct away from the extremal limit where subtleties might arise in the definition of temperature and entropy [25]. The redshift factor R in (3.11) is given by

$$R = \frac{t_0}{t} = \sqrt{g_{00}(0)} = e^{\phi(0)}, \tag{3.14}$$

where $\phi(0)$ is defined in terms of the Feynman diagrams given in figure 4. We now calculate these diagrams to obtain R.

The first two diagrams, figure 4(a) and (b) are charge independent and their value is given by [12]

$$\text{Figure } 4(a) = -\lambda, \tag{3.15}$$

$$\text{Figure } 4(b) = +\lambda^2. \tag{3.16}$$

Evidently, the computation of the redshift factor is only modified at $\mathcal{O}(\lambda^2)$ by diagram figure 4(c). The steps to compute this additional diagram are similar to those used to compute figure 2(d). It is merely obtained by dividing the result of figure 2(d) by $-m_0$. This in turn yields

$$\text{Figure } 4(c) = -\frac{Q^2}{4m_0^2} \frac{d-2}{d-3} \lambda^2. \tag{3.17}$$

Combining the above results, we obtain the redshift factor

$$R = 1 - \lambda + \left(1 + \frac{\eta^2}{2}\right) \lambda^2 + \dots. \tag{3.18}$$

In the extremal limit $\eta = 0$, the redshift factor agrees with the exact solution constructed in section 4. For the temperature to $\mathcal{O}(\lambda^2)$ in the case of a charged compactified BH the result is then

$$T = T_0 \left[1 - \lambda + \left(1 + \frac{\eta^2}{2}\right) \lambda^2 + \dots\right]. \tag{3.19}$$

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One can close the thermodynamic equations by considering the entropy of the charged caged BH. Since the contribution of finite size operators are always beyond $\mathcal{O}(\lambda^2)$ in the perturbative expansion, we can deduce horizon deformations due to the presence of the compact dimension can be neglected. In other words, the BH horizon is unaltered by the compactified dimension to $\mathcal{O}(\lambda^2)$ and its area is fixed by the values of m_0 and Q only. Since the entropy of the BH is proportional to the area of the horizon, we can write down the entropy of the charged caged BH to $\mathcal{O}(\lambda^2)$ as follows

$$S = S_0(1 + 0\,\lambda + 0\,\lambda^2 + \ldots), \tag{3.20}$$

where S_0 is the entropy of an uncompactified d-dimensional charged BH given by (3.13). In appendix B we present an independent derivation of the thermodynamic relations using the Helmholtz free energy, which does not use the uncompactified entropy as an input.

From the first law of thermodynamics

$$dm = TdS + \Phi dQ + \hat{\tau}dL, \tag{3.21}$$

one can derive the expression for the electrostatic potential

$$\Phi = \left(\frac{\partial m}{\partial Q}\right)_{S,L} = \frac{\partial(m, S, L)}{\partial(Q, m_0, L)}\left(\frac{\partial(Q, S, L)}{\partial(Q, m_0, L)}\right)^{-1}$$
$$= \left(\frac{\partial m}{\partial Q}\right) - \left(\frac{\partial m}{\partial m_0}\right)\left(\frac{\partial S}{\partial Q}\right)\left(\frac{\partial S}{\partial m_0}\right)^{-1}, \tag{3.22}$$

where we have used the standard change of thermodynamic variables in terms of Jacobians [26]. Substituting explicit expressions for S and m from (3.7) and (3.13) respectively, we obtain

$$\Phi = \Phi_0\left[1 + \eta\,\lambda - \frac{\eta}{2}(2 - \eta)\lambda^2 + \ldots\right], \tag{3.23}$$

where $\Phi_0 = (m_0/Q)(1 - \eta)$.

To compute the tension $\hat{\tau}$, we use the Smarr relation

$$(d - 3)(m - \Phi Q) = (d - 2)TS + \hat{\tau}L, \tag{3.24}$$

which gives

$$\frac{\hat{\tau}L}{m_0} = \frac{1}{2}(d - 3)\eta^2\lambda - (d - 3)\eta^2\lambda^2 + \ldots. \tag{3.25}$$

We have explicitly checked this result for the tension via a direct diagrammatic computation with an external $\bar{\sigma}_{zz}$, see section 5 for further details on the procedure.

The Gibbs free energy

$$G = G(T, \Phi, L) = m - TS - \Phi Q, \tag{3.26}$$

which implies $dG = -SdT - Qd\Phi + \hat{\tau}dL$. Although we did not need to use the Gibbs free energy above, all the thermodynamics can be derived from G by forming the appropriate differentials. The result reads

$$G(m_0, Q, L) = G_0\left[1 + \left(\frac{d-2}{2}\eta - 1\right)\lambda + \left(1 - (d-2)\eta + \frac{\eta^2}{2}\right)\lambda^2 + \ldots\right], \tag{3.27}$$

where $G_0 = \eta\, m_0/(d-2)$ is the Gibbs potential of a charged BH in the uncompactified spacetime $\mathbb{R}^{d-1,1}$.

Examining our thermodynamic calculations, we recover the results for an uncharged caged BH of [8, 12] in the vanishing charge limit $\eta = 1$. On the other hand, when the extremality condition $\eta = 0$ holds, we have found to $\mathcal{O}(\lambda^2)$ the mass and the electrostatic potential remain unrenormalized and the redshift R is given by (3.18) with $\eta = 0$. In addition, according to (3.25), the tension of an extremally charged BH vanishes to second order identically. Physically, this result states that electrostatic repulsion exactly cancels gravitational attraction to this order in λ. These results are in agreement with the exact solution we construct for an extremally caged BH in the next section.

4   Charged caged BHs in the extremal limit

It has been known for some time that Einstein-Maxwell theory admits static multi-BH solutions [27]. The existence of these solutions can be attributed to the balance of electromagnetic and gravitational forces when the extremal condition (3.9) is satisfied. In this section we follow [28] and [29] and build the thermodynamics for our extremally charged caged BH. This will allow us to verify the perturbative expansions of the previous section in the extremal limit.

4.1   Exact solution

The action we employ is the Euclidean Einstein-Maxwell action[1] supplemented with the Gibbons-Hawking boundary term [28]. This additional term is included to obtain an action depending only on the first derivatives of the metric. We therefore have

$$S_E^{\text{full}}[g_{\mu\nu}^f, \mathcal{A}_\mu] = \frac{-1}{16\pi G} \int d^d x \sqrt{g_f} \left(\mathcal{R}[g_{\mu\nu}^f] - \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} \right) - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{h}\,(\mathcal{K} - \mathcal{K}_0), \quad (4.1)$$

where $g_{\mu\nu}^f$ and $\mathcal{A}_\mu$ are the full metric and the full electromagnetic d-vector potential including all length scales involved in the problem; $\mathcal{R}$ is the Ricci scalar and $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ is the field strength tensor, h_{ij} represents an induced metric on $\partial\mathcal{M}$, and $\mathcal{K}$ and $\mathcal{K}_0$ are the trace of the second fundamental form of the boundary for a particular metric and a flat metric respectively.

For the caged BH setup, the gravitational field can be thought to originate from the BH and its associated images in the covering space. In other words, the setup is equivalent to the case in which an infinite number of identical equidistant BHs lie on a common axis (the z-axis) of a d-dimensional uncompactified spacetime, see figure 5. This arrangement has the following solution [29]

$$ds^2 = g_{\mu\nu}^f dx^\mu dx^\nu = H^{-2} d\tau^2 + H^{2/(d-3)} \left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\Omega_{d-3}^2 \right), \quad (4.2)$$

---

[1]In this section we change our notation to emphasize that we are not working in the EFT but in full GR, namely all scales are present at all times. Also, since our problem is static, switching between Lorenzian and Euclidean signatures presents no difficulties and is given by $t \to it$.
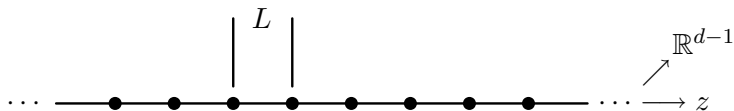
Figure 5. Schematic of exact solution construction.

where H extremizes the action (4.1) and is given below, $d\Omega_{d-3}^2$ denotes the metric on the unit $(d-3)$-sphere and τ corresponds to the Euclidean time. The Maxwell field strength is $\mathcal{F} = d\mathcal{A}$, where

$$\mathcal{A} = -\sqrt{\frac{d-2}{2(d-3)}}\, \text{sign}(Q)\, H^{-1} d\tau. \tag{4.3}$$

The function H is harmonic on $\mathbb{R}^{d-1}$, with poles corresponding to the locations of the event horizons of the BHs and is given by

$$H = 1 + \frac{r_0^{d-3}}{2} \sum_{n=-\infty}^{\infty} \frac{1}{(\mathbf{x}_\perp^2 + (z - nL)^2)^{(d-3)/2}}, \tag{4.4}$$

where $|\mathbf{x}_\perp| = \rho \sin\theta$ and $z = \rho \cos\theta$.

A comment should be made regarding the gauge choice of the 1-form $\mathcal{A}$. As can be seen from (4.3), $\mathcal{A}_0 = 0$ at any event horizon. Since $d\tau$ is not defined on the horizon, this choice of gauge avoids the possibility of a singularity in the 1-form at any event horizon. The potential at infinity does not vanish in this gauge, so the electrostatic potential is given by

$$\Phi = \mathcal{A}_0(\rho = 0) - \mathcal{A}_0(\infty) = \sqrt{\frac{d-2}{2(d-3)}}\, \text{sign}(Q)\,, \tag{4.5}$$

where we have chosen to evaluate Φ at the horizon located at the origin.

If $L \to \infty$ in (4.4), so that only the $n = 0$ term remains, we recover the standard Reissner-Nordström extremal solution written in the isotropic radial coordinate ρ. Its relation to the Schwarzschild coordinate r is given by

$$\rho = \left(r^{d-3} - \frac{1}{2}\, r_0^{d-3} \right)^{1/(d-3)}. \tag{4.6}$$

4.2 Thermodynamics

In what follows we evaluate the Euclidean action (4.1) on the solutions (4.2) and (4.3). We find the Euclidean action vanishes and this leads to the conclusion that the Gibbs potential of an extremal charged caged BH equals zero, via the relation

$$-\beta G(m_0, L) = \ln Z = -S_E^{\text{full}}[g_{\mu\nu}^f, \mathcal{A}_\mu], \tag{4.7}$$

where β is the inverse Euclidean temperature and Z is the partition function. As shown in [25], due to the topology change of the spacetime manifold, the Euclidean extremal Reissner-Nordström solution has two boundaries: at infinity (as in the non-extremal case)

and at the horizon. We now evaluate the contribution these boundaries make to the Euclidean action.

We first consider the boundary at infinity and we choose $r_\perp = const$, where $r_\perp = |\mathbf{x}_\perp|$, to represent this surface. This boundary is compact and has the topology $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^{d-3}$. The series in (4.4) can be summed explicitly if d is odd, that is $d = 2j + 5, j = 0, 1, 2, \ldots$. In this case

$$
\begin{aligned}
H &= 1 + \frac{r_0^{d-3}}{2} \frac{(-1)^j}{j!} \frac{d^j}{d(\mathbf{x}_\perp^2)^j} \sum_{n=-\infty}^{\infty} \frac{1}{\mathbf{x}_\perp^2 + (z - nL)^2} \\
&= 1 + r_0^{d-3} \frac{\pi}{2L} \frac{(-1)^j}{j!} \frac{d^j}{d(\mathbf{x}_\perp^2)^j} \left[\frac{1}{|\mathbf{x}_\perp|} \frac{\sinh\left(\frac{2\pi|\mathbf{x}_\perp|}{L}\right)}{\cosh\left(\frac{2\pi|\mathbf{x}_\perp|}{L}\right) - \cos\left(\frac{2\pi z}{L}\right)} \right] \\
&= 1 + \frac{d-3}{2(d-4)} \frac{\Omega_{d-2}}{\Omega_{d-3}} \frac{r_0}{L} \left(\frac{r_0}{|\mathbf{x}_\perp|}\right)^{d-4},
\end{aligned}
\tag{4.8}
$$

where in the last equality we suppressed exponentially small terms of order $\sim \exp(-L/|\mathbf{x}_\perp|)$. These terms do not contribute further in our computations. Further, since the leading order term contains no information about parity, one can show this result is valid for any d. Therefore we have

$$
\begin{aligned}
ds^2 &= \left(1 + \frac{d-3}{2(d-4)} \frac{\Omega_{d-2}}{\Omega_{d-3}} \frac{r_0}{L} \left(\frac{r_0}{r_\perp}\right)^{d-4}\right)^{-2} d\tau^2 \\
&\quad + \left(1 + \frac{d-3}{2(d-4)} \frac{\Omega_{d-2}}{\Omega_{d-3}} \frac{r_0}{L} \left(\frac{r_0}{r_\perp}\right)^{d-4}\right)^{2/(d-3)} \left(dz^2 + dr_\perp^2 + r_\perp^2 d\Omega_{d-3}^2\right), \\
\mathcal{F} &= d\mathcal{A} = -\sqrt{\frac{(d-2)(d-3)}{8}} \frac{1}{L} \frac{\Omega_{d-2}}{\Omega_{d-3}} \left(\frac{r_0}{r_\perp}\right)^{d-3} H^{-2} dr_\perp \wedge dt.
\end{aligned}
\tag{4.9}
$$

One can evaluate the corresponding Gibbons-Hawking boundary term relying on the identity

$$
\int_{\partial\mathcal{M}} \sqrt{h} \mathcal{K} = \frac{\partial}{\partial n} \int_{\partial\mathcal{M}} \sqrt{h},
\tag{4.10}
$$

where $(\partial/\partial n) \int_{\partial\mathcal{M}} \sqrt{h}$ is the derivative of the area of $\partial\mathcal{M}$ as each point of the boundary is moved an equal distance along the outward unit normal n. Thus in our case we get

$$
\begin{aligned}
\int_{r_\perp=const} \sqrt{h} \, \mathcal{K} &= H^{-1/(d-3)} \frac{d}{dr_\perp} \left(\beta L \, \Omega_{d-3} H^{1/(d-3)} r_\perp^{d-3}\right) \\
&= \beta L \, \Omega_{d-3} \left[(d-3) r_\perp^{d-4} - \frac{\Omega_{d-2}}{\Omega_{d-3}} \frac{r_0^{d-3}}{2L}\right].
\end{aligned}
\tag{4.11}
$$

For flat space $\mathcal{K}_0 = (d-3) H^{-1/(d-3)} r_\perp^{-1}$, therefore

$$
\int_{r_\perp=const} \sqrt{h} \, \mathcal{K}_0 = \beta L \, \Omega_{d-3}(d-3) r_\perp^{d-4}.
\tag{4.12}
$$

Combining these relations together yields

$$\frac{1}{8\pi G} \int_{r_\perp = const} \sqrt{h}\, (\mathcal{K} - \mathcal{K}_0) = -\frac{\beta m_0}{d - 2}. \tag{4.13}$$

We now turn to the boundary at the horizon, given by $\rho = 0$. To compute this boundary contribution, we consider the hypersurface $\rho = const$ and then take the limit $\rho \to 0$ at the end of the calculation. An outward-pointing normal vector for such a hypersurface is $n_\mu = -\sqrt{g_{\rho\rho}}\, \delta_{\mu\rho}$ (no sum on ρ) and the induced metric $h_{\mu\nu}$ is given by

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu = H^{-2} d\tau^2 + H^{2/(d-3)} \rho^2 \left(d\theta^2 + \sin^2\theta\, d\Omega_{d-3}^2 \right), \tag{4.14}$$

and the covariant derivative of n_μ is

$$n_{\nu;\,\mu} = -\sqrt{g_{\rho\rho}} \left(\delta_{\mu\rho}\, \delta_{\nu\rho}\, \partial_\rho \ln \sqrt{g_{\rho\rho}} + \delta_{\mu\theta}\, \delta_{\nu\rho}\, \partial_\theta \ln \sqrt{g_{\rho\rho}} - \Gamma^\rho_{\mu\nu} \right). \tag{4.15}$$

We can now evaluate the trace of the second fundamental form on the boundary

$$\mathcal{K} = h^{\nu\mu}\, n_{\nu;\,\mu} = -H^{-1/(d-3)} \left(\frac{d-2}{\rho} + \frac{1}{d-3} \frac{\partial_\rho H}{H} \right). \tag{4.16}$$

From the definition (4.4) of H, we get the following expansion in the vicinity of the horizon

$$H = \frac{1}{2} \left(\frac{r_0}{\rho} \right)^{d-3} \left(1 + \mathcal{O}(\rho^{d-3}) \right),$$

$$\frac{\partial_\rho H}{H} = \frac{3-d}{\rho} \left(1 + \mathcal{O}(\rho^{d-2}) \right). \tag{4.17}$$

The regulator in this case is given by $\mathcal{K}_0 = -(d-2)H^{-1/(d-3)}/\rho$ and we therefore obtain

$$\frac{1}{8\pi G} \int_{\rho = const} \sqrt{h}\, (\mathcal{K} - \mathcal{K}_0) = \frac{\beta\, \Omega_{d-2}}{8\pi G} \rho^{d-3} \left(1 + \mathcal{O}(\rho^{d-2}) \right). \tag{4.18}$$

As a result, the horizon boundary term vanishes when ρ tends to zero, that is

$$\frac{1}{8\pi G} \int_{\rho=0} \sqrt{h}\, (\mathcal{K} - \mathcal{K}_0) = 0. \tag{4.19}$$

In order to complete the calculation of the Euclidean action, one has to compute the Einstein-Maxwell terms in (4.1). Since the scalar curvature $\mathcal{R}$ can be evaluated from the Einstein field equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G\, \mathcal{T}_{\mu\nu} \Rightarrow \ \mathcal{R} = \frac{d-4}{d-2} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}, \tag{4.20}$$

where

$$\mathcal{T}_{\mu\nu} = \frac{1}{4\pi G} (\mathcal{F}_\mu{}^\alpha \mathcal{F}_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} \mathcal{F}^{\alpha\beta} \mathcal{F}_{\alpha\beta}), \tag{4.21}$$

is the energy-momentum tensor of the Maxwell field, it is enough to evaluate the electromagnetic part of the action.

For this purpose, recall that for a solution of the Maxwell equations $\mathcal{F}^{\mu\nu}{}_{;\nu} = 0$, one can rewrite the Maxwell field Lagrangian density as $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = (2\mathcal{F}^{\mu\nu}\mathcal{A}_\mu)_{;\nu}$. Combining with (4.9) gives

$$\frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{g}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = \frac{1}{8\pi G}\int \mathcal{F}^{\mu\nu}\mathcal{A}_\mu d\Sigma_\nu = -\frac{\beta m_0}{2}. \tag{4.22}$$

We are now in a position to compute the Gibbs free energy. Substituting (4.13), (4.19), (4.20), and (4.22) into (4.1) yields

$$-\beta G(m_0, L) = -S_E^{\text{full}} = 0. \tag{4.23}$$

Since $G(m_0, L) = m - TS - \Phi Q$ this implies

$$m - \Phi Q = TS, \tag{4.24}$$

where m is the total energy (mass) of the system. However, by the generalized Smarr's relation $(d-3)(m - \Phi Q) = (d-2)TS + \hat{\tau}L$. Thus $-TS = \hat{\tau}L$. Since all the quantities involved in the last equality are positive definite and L does not vanish, we conclude in the extremal case

$$TS = \hat{\tau} = 0, \tag{4.25}$$

so the tension of the system is zero. Substituting this result back into (4.24) and using the extremality condition (3.9), we see the mass of the system is unrenormalized

$$m = \Phi Q = m_0. \tag{4.26}$$

There remains one last thermodynamic quantity of interest. The redshift factor (3.14), is easily computed from our exact extremal solution and we have

$$\sqrt{g_{00}^f(\mathbf{x})} = H^{-1}(\mathbf{x}) \Rightarrow R = \sqrt{g_{00}(O)} = (1 + \lambda)^{-1}, \tag{4.27}$$

where in the last equality we excluded the $n = 0$ term in the definition of H, since it corresponds to the short wavelength scale of order r_0. This term is a pure infinity and is also present in the EFT approach, but in that case it is set to zero via dimensional regularization. For our purposes here, it can be ignored and thus does not contribute to the value of g_{00}. Expanding the last equation in λ, one recovers (3.18) for $\eta = 0$.

The results in (4.25), (4.26), and (4.27) summarize the major outcomes of this section. In particular, these results verify the non-renormalization statements obtained perturbatively in section 3 via the EFT approach.

5 Finite size effects

In this section we add non-minimal couplings to account for finite size effects and study the resulting thermodynamics. We start with the uncharged case and then proceed to charged caged BHs. We determine the order the non-minimal operators enter the perturbative expansion for charged BHs. The existence of the exact solution in the extremal case allows us to undertake a matching calculation for the Wilson coefficients of the finite size operators in this limit.

5.1 Uncharged caged BHs

To incorporate the finite size operators into the action, we need to add all possible non-minimal operators which respect diffeomorphism and reparametrization invariance. These operators must use proper time throughout and can involve combinations of the geometric invariants, i.e. the Ricci scalar R, the Ricci tensor $R_{\mu\nu}$, and the Riemann tensor $R_{\mu\nu\rho\sigma}$. Higher derivative combinations of these terms are also possible.

As usual in an EFT, the derivative expansion is truncated at a given order determined by the accuracy required in the calculation. In the uncharged case [9] it was shown the operators involving R and $R_{\mu\nu}$ can be removed by field redefinitions. These operators are redundant since the vacuum equations of motion are $R_{\mu\nu} = 0$. This means all physical finite size operators can be built from the Weyl tensor $C_{\mu\nu\rho\sigma}$ or alternatively from the Riemann tensor. Therefore, the leading order finite size operators are composed of the electric and magnetic components of the Weyl tensor squared [18]. Hence, the worldline action including these operators becomes

$$S_{\mathrm{BH}}[x, g_{\mu\nu}] = -m_0 \int d\tau + \gamma_1 \int d\tau E_{\mu\nu} E^{\mu\nu} + \delta_1 \int d\tau B_{\mu_1...\mu_{d-2}} B^{\mu_1...\mu_{d-2}}, \qquad (5.1)$$

where

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}, \qquad B_{\mu_1...\mu_{d-2}} = \frac{1}{(d-2)!} \epsilon_{\alpha\mu_1...\mu_{d-3}\gamma\delta} C^{\gamma\delta}{}_{\beta\mu_{d-2}} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}. \qquad (5.2)$$

Since our setup is static, the magnetic components of the Weyl tensor do not contribute to any thermodynamic observable and so we ignore the δ_1 operator. If we consider a non-static problem such as a scattering process, this operator may contribute since its Wilson coefficient δ_1 may be non-zero.

The γ_1 operator will give rise to new vertices coupling to the worldline. The relevant worldline couplings will be $\phi\phi$ and $\phi\phi\bar\sigma_{zz}$. These are derived by expanding the operator and the result is

$$S_{\mathrm{BH}} \supset \gamma_1 (\partial_i \partial_j \phi)^2 - 2\gamma_1 \bar\sigma_{ij} (\partial_i \partial_k \phi)(\partial_j \partial_k \phi), \qquad (5.3)$$

where we have neglected terms which vanish by the leading order equations of motion. Terms with derivatives acting on $\bar\sigma_{ij}$ in (5.3) have also been omitted since these do not contribute to our calculations.

To derive the thermodynamics including finite size effects, we need to use a different approach compared to section 3. There we assumed the local temperature T_0 in (3.11) is given by the temperature of a spherical uncompactified BH (3.12) and derived the asymptotic temperature using the redshift. This methodology is justified when the BH is spherical. However once finite size effects are allowed on the worldline, this is no longer the case. Instead, we must first calculate the mass and the tension of the system. With these relations we can derive the remaining thermodynamic quantities, independent of the BH shape. The redshift is also an observable so we compute it for completeness.

A direct computation of the tension from Feynman diagrams requires we use the prescription of [8] and compute $\int d^{d-1}\mathbf{x} T^{zz} = -\hat\tau L$. We will however use the KK metric
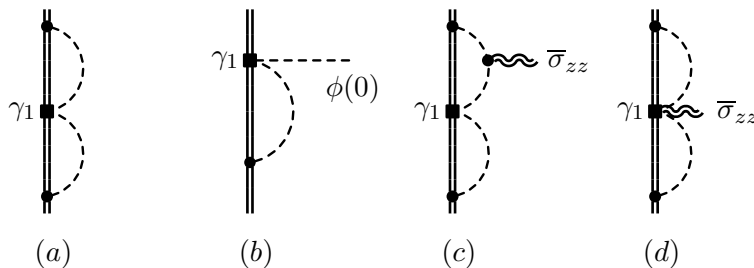
Figure 6. Feynman diagrams with the finite size operator γ_1. (a) Contribution to the ADM mass. (b) Redshift modification. (c) and (d) Tension contributions from γ_1.

parametrization where the tension is obtained from diagrams with one external $\bar{\sigma}_{zz}$. In this case the sum of all diagrams yields $-\frac{1}{2}\hat{\tau}L\bar{\sigma}_{zz}$.

The Feynman diagrams needed to compute the contribution of the γ_1 operator to the mass, the redshift, and the tension are displayed in figure 6. Their evaluation requires some new sum integrals which are listed in appendix A. We obtain

$$\text{Figure } 6(a) = \tilde{\gamma}_1 m_0 (d-1)(d-2)(d-3)^2 \mu^2, \tag{5.4}$$

$$\text{Figure } 6(b) = -2\tilde{\gamma}_1 (d-1)(d-2)(d-3)^2 \mu^2, \tag{5.5}$$

$$\text{Figure } 6(c) = -\tilde{\gamma}_1 m_0 (d-2)(d-3)^2 (d^2 - 4d + 5)\mu^2 \bar{\sigma}_{zz}, \tag{5.6}$$

$$\text{Figure } 6(d) = -2\tilde{\gamma}_1 m_0 (d-2)^2 (d-3)^2 \mu^2 \bar{\sigma}_{zz}, \tag{5.7}$$

where we have defined $\gamma_1 = \tilde{\gamma}_1 m_0 r_0^4$ so $\tilde{\gamma}_1$ is dimensionless and the expansion parameter is defined by

$$\mu = \left(\frac{r_0}{L}\right)^{d-1} \zeta(d-1) \sim \lambda^{\frac{d-1}{d-3}}. \tag{5.8}$$

The $\mathcal{O}(\mu^2)$ corrections to the mass, redshift and tension are then

$$m = m_0 \Big[1 - \tilde{\gamma}_1 (d-1)(d-2)(d-3)^2 \mu^2 \Big], \tag{5.9}$$

$$R = 1 - 2\tilde{\gamma}_1 (d-1)(d-2)(d-3)^2 \mu^2, \tag{5.10}$$

$$\frac{\hat{\tau}L}{m_0} = 2\tilde{\gamma}_1 (d-1)^2 (d-2)(d-3)^2 \mu^2, \tag{5.11}$$

where we did not include the $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^2)$ contributions from section 3. We see that all corrections to thermodynamic quantities from the leading finite size effects scale as $\mu^2 \sim \lambda^{\frac{2(d-1)}{d-3}}$, where $\mu^2 \sim \lambda^4$ for $d = 5$, $\mu^2 \sim \lambda^3$ for $d = 7$ and $\mu^2 \sim \lambda^p$ with $2 < p < 3$ for $d > 7$.

From the Smarr relation and the first law we find the entropy

$$S = S_0 \left[1 + 0 \cdot \tilde{\gamma}_1 \mu^2 \right], \tag{5.12}$$

with S_0 given by (3.13) in the limit $\eta = 1$ and the temperature

$$T = T(L \to \infty) \left[1 - \tilde{\gamma}_1 (3d-5)(d-1)(d-2)(d-3)\mu^2 \right], \tag{5.13}$$

where $T(L \to \infty)$ is the uncompactified temperature of (3.12) with $\eta = 1$. We note that when finite size effects are included, the entropy remains non-renormalized at leading order. Moreover, we observe that the redshift R does not coincide with the renormalization of the temperature — the local temperature T_0 is not simply given by $T(L \to \infty)$. In fact, we can now extract the local temperature including the leading order finite size operator to be

$$\tilde{T}_0 = T/R = T(L \to \infty) \left[1 - \tilde{\gamma}_1 (d+1)(d-1)(d-2)(d-3)\mu^2 \right]. \qquad (5.14)$$

The Wilson coefficient $\tilde{\gamma}_1$ is still undetermined and its exact value must be extracted from a matching calculation at the scale r_0. If possible, any matching calculation is best undertaken in d dimensions since there is no ambiguity when extracting the Wilson coefficients. In d dimensions the finite size contributions to the thermodynamics have a different dimensional scaling compared to the point particle contributions. However, this is beyond the scope of the current work. Since $\tilde{\gamma}_1$ is dimensionless, one would expect $\tilde{\gamma}_1 \sim \mathcal{O}(1)$; however, it can depend on the dimension d.

5.2 Including charge

For a charged caged BH, we do not have a Ricci flat background, so we need to consider the finite size operators with two derivatives. In general, the effective BH action (2.4) including all terms up to two derivatives is

$$\begin{aligned}
S_{\mathrm{BH}}[x_\mu, g_{\mu\nu}, a_\mu] = &-m_0 \int d\tau + Q \int dx^\mu a_\mu \\
&+\alpha_1 \int d\tau R + \alpha_2 \int d\tau R_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \\
&+\alpha_3 \int d\tau f_{\mu\nu} f^{\mu\nu} + \alpha_4 \int d\tau f_{\mu\sigma} f_\nu^\sigma \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \ldots, \qquad (5.15)
\end{aligned}$$

where we have restored the time integrals for convenience and in the last two lines we have introduced the lowest order operators describing finite size effects. According to the power counting rules [8, 12], the Wilson coefficients of these operators scale as $m_0 r_0^2$ if we power count $Q \sim m_0$.

Due to the presence of electromagnetism, operators involving the Ricci tensor with coefficients α_1 and α_2 do not vanish by the leading order equations of motion. However we can still use the equations of motion, where $R \sim ff$ for the Ricci tensor and scalar, see (4.20). Thus we can eliminate the α_1 and α_2 operators in favor of the α_3 and α_4 operators. Alternatively, it is also possible to redefine the coordinate system to eliminate α_3 and α_4, and the procedure is similar to the uncharged case discussed in [9].

We will eliminate the α_1 and α_2 operators and use a different operator basis for the electromagnetic finite size operators than in (5.15) where the α_3 and α_4 operators are expressed in terms of the electric e_μ and the magnetic $b_{\mu_1 \ldots \mu_{d-3}}$ components of the Maxwell field strength tensor

$$\begin{aligned}
e_\mu &= f_{\nu\mu} \frac{dx^\nu}{d\tau}, \\
b_{\mu_1 \ldots \mu_{d-3}} &= \frac{1}{(d-3)!} \frac{dx^\alpha}{d\tau} \epsilon_{\alpha \mu_1 \ldots \mu_{d-3} \beta\gamma} f^{\beta\gamma} . \qquad (5.16)
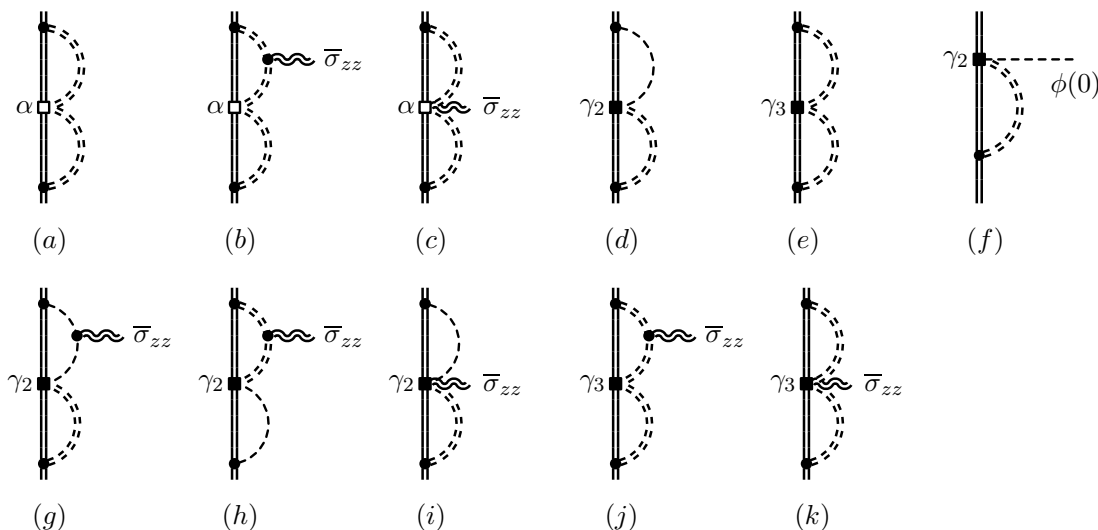\end{aligned}$$

Figure 7. (a) Contribution to the ADM mass m from the finite size operator insertion. The open square denotes the vertex resulting from $\alpha \int d\tau e_\mu e^\mu$ insertion. (b) and (c) Tension contributions from α operator. (d) and (e) ADM mass renormalization diagrams resulting from the quadrupole finite size operators. The solid square vertex denotes the appropriate quadrupole finite size operators. (f) Contribution to the redshift of operator γ_2. (g)-(k) Tension contributions from the γ_2 and γ_3 finite size operators.

For a static system the operator formed from the magnetic field will not contribute to the thermodynamics, so we will neglect it. The worldline action including the leading order static finite size operator with two derivatives reads

$$S_{\rm BH}[x_\mu, g_{\mu\nu}, a_\mu] = -m_0 \int d\tau + Q \int dx^\mu a_\mu + \alpha \int d\tau e_\mu e^\mu + \dots . \qquad (5.17)$$

We now compute the effect of the α operator. In the BH's rest frame the α operator reduces to $-e^{2\phi/(d-3)} \gamma^{ij} (\partial_i \varphi)(\partial_j \varphi)$. The Feynman rule for the vertex coupling two φ's to the worldline is $2\alpha \mathbf{k} \cdot \mathbf{p}$, with both momenta incoming. The corresponding mass renormalization diagram is shown in figure 7(a), which scales as $\sim m_0 \lambda^{2(d-2)/(d-3)}$. Calculating this diagram shows that it vanishes, and this has a clear interpretation. The finite size operator in (5.17) encodes deformation due to the electric dipole polarization of the BH. If we think of the problem as a line of charged BHs at regular intervals in the covering space, as in figure 5, then the polarizing effect from a charge to the right will be canceled by the equidistant charge on the left. Therefore the overall polarizing effect is zero and there is no mass renormalization from this worldline operator. Similarly, the diagrams involving this operator and an external $\bar{\sigma}_{zz}$ which contribute to the tension, see figure 7(b) and (c), are found to vanish.

As a result, the first non-vanishing finite size contributions to the BH thermodynamics come from operators with four derivatives, which encode deformations due to induced quadrupole moments. As explained above, in a static system we can form all relevant operators using the electric field e_μ, the electric components of the Weyl tensor $E_{\mu\nu}$, and higher derivative combinations. The static finite size terms in the action read at order four

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derivatives

$$S_{\text{BH}}[x, g_{\mu\nu}, a_\mu] = \gamma_1 \int d\tau \, E_{\mu\nu} E^{\mu\nu} - \gamma_2 \int d\tau (\nabla_\mu e_\nu) E^{\mu\nu} + \gamma_3 \int d\tau (\nabla_\mu e_\nu)(\nabla^\mu e^\nu), \quad (5.18)$$

where the first operator coincides with the one in the uncharged case of (5.1).

The finite size terms in the action (5.18) lead to new worldline vertices with two scalar fields, either gravitational or electromagnetic. There are also vertices on the worldline with two scalars and one $\bar{\sigma}_{zz}$ which contributes to the tension calculation. The leading order correction to the ADM mass due to an insertion of these operators introduced in (5.18) arises from the diagrams figure 6(a) and figure 7(d) and (e). Calculating the new diagrams gives the renormalized mass

$$m = m_0 \left(1 - \left[\tilde{\gamma}_1 + \tilde{\gamma}_2 \frac{1}{2} \frac{Q}{m_0} \frac{d-2}{d-3} + \tilde{\gamma}_3 \frac{1}{4} \left(\frac{Q}{m_0} \right)^2 \left(\frac{d-2}{d-3} \right)^2 \right] (d-1)(d-2)(d-3)^2 \mu^2 \right), \tag{5.19}$$

where we defined the dimensionless couplings $\tilde{\gamma}_i$ from $\gamma_i = \tilde{\gamma}_i m_0 r_0^4$.

For the redshift factor R we compute figure 6(b) and figure 7(f) with the result

$$R = 1 - \left(2\tilde{\gamma}_1 + \tilde{\gamma}_2 \frac{1}{2} \frac{Q}{m_0} \frac{d-2}{d-3} \right) (d-1)(d-2)(d-3)^2 \mu^2, \tag{5.20}$$

and the tension is computed via the diagrams figure 6(c) and (d) and figure 7(g)-(k) to be

$$\frac{\hat{\tau} L}{m_0} = 2 \left[\tilde{\gamma}_1 + \tilde{\gamma}_2 \frac{1}{2} \frac{Q}{m_0} \frac{d-2}{d-3} + \tilde{\gamma}_3 \frac{1}{4} \left(\frac{Q}{m_0} \right)^2 \left(\frac{d-2}{d-3} \right)^2 \right] (d-1)^2 (d-2)(d-3)^2 \mu^2. \tag{5.21}$$

In the charged case we have two dimensionful short distance quantities, m_0 and Q, so in general the dimensionless couplings $\tilde{\gamma}_i$ are functions of the order parameter η. Thus the thermodynamics derivation of the remaining quantities S, T and Φ becomes cumbersome and not insightful since we have to keep derivatives of the couplings with respect to Q and m_0.

In the extremal limit $\eta = 0$, we can use the exact solution of section 4 to undertake a matching calculation for the undetermined Wilson coefficients. This is done by comparing the exact results $m = m_0$, $\hat{\tau} = 0$, and $R = (1 + \lambda)^{-1}$ to the expressions in the perturbative expansions (5.19), (5.20), and (5.21). The non-renormalization of the mass and tension are seen to yield one relation between the Wilson coefficients $\tilde{\gamma}_i$, since the linear combination of coefficients in (5.19) and (5.21) is degenerate.

The remaining relation is derived from R. We note the $\mathcal{O}(\mu^2)$ contribution must vanish, since it is non-analytic in λ in arbitrary dimensions. We find from (5.20)

$$\left[\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1} \right]_{\eta=0} = -\sqrt{8 \frac{d-3}{d-2}} \, \text{sign}(Q). \tag{5.22}$$

Using this relation and $m = m_0$ or $\hat{\tau} = 0$ we get

$$\left[\frac{\tilde{\gamma}_3}{\tilde{\gamma}_1} \right]_{\eta=0} = 2 \frac{d-3}{d-2}. \tag{5.23}$$

This constitutes a partial matching of the Wilson coefficients of the finite size operators in the extremal limit.

The matching calculation presented above has several shortcomings. The calculation of the thermodynamic properties of caged BHs cannot yield the coefficient of any operator involving magnetic type components, since these do not contribute for static systems. Further, even in the case of the electric type operators, the leading finite size operator with two derivatives $e_\mu e^\mu$ does not contribute to the thermodynamics of caged BHs — so we cannot fix its Wilson coefficient α. Lastly, while the three electric type operators with four derivatives do contribute to the thermodynamics, we are not able to fix all three Wilson coefficients γ_i uniquely, due to the degeneracy between the mass and tension relationships.

We have not exploited the non-renormalization of the electrostatic potential Φ. We might may be able to fix the Wilson coefficients γ_i uniquely if we could calculate the electrostatic potential Φ in the EFT. If we succeeded in such a computation, the $\mathcal{O}(\mu^2)$ contributions from the finite size operators would most likely entail a linear combination of γ_2 and γ_3. If this new equation was linearly independent of the other equations derived here, this would imply $\gamma_1 = \gamma_2 = \gamma_3 = 0$ in the extremal limit.

6 Conclusion

Using an EFT approach we have analyzed the thermodynamic properties of small compactified BHs carrying charge. We obtain the relevant thermodynamic quantities to $\mathcal{O}(\lambda^2)$. Standard power counting arguments show that up to $\mathcal{O}(\lambda^2)$, all thermodynamic contributions arise from a point particle description of the BH. This implies that to this order the horizon is spherical and not deformed due to the presence of the compact dimension, and the entropy is given by the area of a spherical horizon as for an uncompactified charged BH.

In the extremal limit, we constructed an exact solution using the standard methods of GR. From the exact solution, we find that the mass m, tension τ, and electrostatic potential Φ are non-renormalized in the extremal limit, and the redshift is given by a geometric series in λ. We use the extremal thermodynamics and the uncharged perturbative results of [8, 12] as checks of our perturbative results for the thermodynamic properties and find agreement.

The leading finite size corrections to thermodynamic properties are computed in both the charged and uncharged cases in terms of Wilson coefficients of higher order operators. These coefficients need to be determined by a matching calculation. In the uncharged case the entropy does not acquire a correction due to the leading order finite size operators. If the Wilson coefficient is non-zero, this is interesting, since this means its area remains the same even though the horizon is now deformed and no longer spherical. Moreover, the local temperature and the temperature of an uncompactified BH do not coincide when we include finite size effects, which is a clear sign of a non-sphericity.

For charged caged BHs, we find the leading finite size operators with two derivatives do not contribute to the thermodynamics. Also, due to the presence of two short distance scales Q and m_0, the dimensionless Wilson coefficients can plausibly depend on their ratio. We computed the leading finite size corrections to m, R, and $\hat{\tau}$, which come from

four derivative operators. In the extremal limit, we undertake a matching calculation by comparing with the exact solution and we constrain two out of the three Wilson coefficients.

Our results contribute to the study of the phase diagram for black objects with one compactified dimension. In particular, for dimensions $d > 7$, the leading finite size operators yield the dominant correction to the existing perturbative results at $\mathcal{O}(\lambda^2)$. It would be interesting to perform a complete matching of the Wilson coefficients of the finite size operators. For the uncharged case, it may in fact be possible to obtain an estimate for γ_1 from numerical data for $d > 7$.

In the uncharged case, [8] performed an interesting comparison between their perturbative results and the numerical results of [4]. This comparison shows the perturbative result at $\mathcal{O}(\lambda)$ matches the numerical solution to 10-20% all the way up to the point where the BH/non-uniform BS phase transition occurs. Since we have shown the finite size effects first enter at the same order as in the uncharged case, we can expect our leading order results to behave similarly. We can only make general inferences at this point, since the details of a possible transition are not captured by the EFT.

Based on our result for the leading order tension $\hat{\tau} \propto \eta^2 \lambda$, for a given bare mass m_0 and compactification L, the addition of charge to the caged BH will reduce the tension. Due to the excellent agreement between the EFT and numerical treatments of the uncharged caged BH, we speculate that this property will also hold non-perturbatively. Therefore one can reasonably expect the topology changing transition to a BS phase to be delayed when charge is added to the BH.

Intuitively this increased stability of the BH phase can be understood in the covering space. Here the transition to a BS will occur when we increase the BH mass and the horizons of neighboring image BHs overlap. If there was no horizon deformation this would occur at values of the bare mass when the horizon radius is $r = L/2$. However, the attraction between the neighboring image BHs can cause this overlap to occur at lower bare mass due to horizon deformation. Since like charges repel, the tension from the gravitational attraction will be reduced by an electrostatic repulsion of the image BHs — making a charged caged BH relatively more stable than an uncharged caged BH with the same bare mass.

So far there have been few studies of the BH/BS phase transition with the inclusion of charge. We have covered the entire mass-charge phase space for small λ and our study provides a first exploration of the BH side of the phase space. Due to the presence of the additional parameter Q, a richer structure of the phase diagram can be expected. Further study, either numerically or analytically, towards completion of this phase diagram, is left for future work. Investigation of the possible phases in the extremal or near extremal limit may exhibit new phenomenology.

Acknowledgments

A   Table of Feynman integral sums

All integrals are derived using the techniques described in [30], with the sums being Riemann zeta functions. The first two results listed below, $I_0(L)$ and $I_1(L)$, were derived in [8] and $I_2(L)$ was computed in [12]. We list all integral sums used in this work below

$$I_0(L) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}} \frac{1}{\mathbf{p}_\perp^2 + (2\pi n/L)^2} = \frac{1}{16\pi G m_0} \frac{d-2}{d-3} \lambda, \tag{A.1}$$

$$I_1(L) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}} \left[\frac{2\pi n/L}{\mathbf{p}_\perp^2 + (2\pi n/L)^2} \right]^2 = \left(2 - \frac{d}{2} \right) I_0(L), \tag{A.2}$$

$$I_2(L) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}} \frac{\mathbf{p}_\perp^2}{\mathbf{p}_\perp^2 + (2\pi n/L)^2} = \frac{(d-2)(d-3)}{L^2} \frac{\zeta(d-1)}{\zeta(d-3)} I_0(L), \tag{A.3}$$

$$I_3(L) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}} \frac{(2\pi n/L)^2}{\mathbf{p}_\perp^2 + (2\pi n/L)^2} = -I_2(L), \tag{A.4}$$

$$I_4(L) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}} \frac{(2\pi n/L)^2 \, \mathbf{p}_\perp^2}{\left[\mathbf{p}_\perp^2 + (2\pi n/L)^2 \right]^2} = -\frac{1}{2}(d-2) I_2(L), \tag{A.5}$$

$$I_5(L) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}} \frac{(2\pi n/L)^4}{\left[\mathbf{p}_\perp^2 + (2\pi n/L)^2 \right]^2} = \frac{1}{2}(d-4) I_2(L), \tag{A.6}$$

$$I_6(L) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}\mathbf{p}_\perp}{(2\pi)^{d-2}} \frac{(\mathbf{p}_\perp^2)^2}{\left[\mathbf{p}_\perp^2 + (2\pi n/L)^2 \right]^2} = \frac{d}{2} I_2(L). \tag{A.7}$$

B   Thermodynamics via the Helmholtz free energy

Here we demonstrate the calculation of the thermodynamics of section 3.2 using the Helmholtz free energy F, without using any assumptions about the entropy S of the charged caged BH. We start by defining the Helmholtz free energy

$$F = m - TS. \tag{B.1}$$

The first law (3.21) gives the differential relation $dF = -SdT + \Phi dQ + \hat{\tau} dL$. We now derive the following differential relations between the Helmholtz free energy and the entropy S, the electromagnetic potential Φ, and the tension $\hat{\tau}$,

$$S = -\left(\frac{\partial F}{\partial T} \right)_{Q,L} = -\left(\frac{\partial F}{\partial m_0} \right)_{L,Q} \left[\left(\frac{\partial T}{\partial m_0} \right)_{L,Q} \right]^{-1}, \tag{B.2}$$

$$\Phi = +\left(\frac{\partial F}{\partial Q} \right)_{L,T} = -\frac{\partial(F,L,T)}{\partial(m_0,L,Q)} \left[\left(\frac{\partial T}{\partial m_0} \right)_{L,Q} \right]^{-1}, \tag{B.3}$$

$$\hat{\tau} = +\left(\frac{\partial F}{\partial L} \right)_{T,Q} = +\frac{\partial(F,Q,T)}{\partial(m_0,L,Q)} \left[\left(\frac{\partial T}{\partial m_0} \right)_{L,Q} \right]^{-1}. \tag{B.4}$$

After the second equality in the above relations, we have rewritten the thermodynamic relations to be functions of m_0, L, and Q, since these are the bare parameters of the EFT.

At this stage we use the Smarr relation (3.24) to rewrite F eliminating the temperature-entropy term. This gives

$$(d-2)F = m + (d-3)\Phi Q + \hat{\tau} L. \tag{B.5}$$

This form is now amendable to solution, since we can form a differential relation for F through the above equations for Φ and τ, $(B.3)$ and $(B.4)$ respectively. Doing this allows us to rewrite $(B.5)$ as follows

$$(d-2)F = m + \left((3-d)\,Q\frac{\partial(F,L,T)}{\partial(m_0,L,Q)} + L\,\frac{\partial(F,Q,T)}{\partial(m_0,L,Q)} \right) \left[\left(\frac{\partial T}{\partial m_0} \right)_{L,Q} \right]^{-1}. \tag{B.6}$$

We have now obtained what is obviously a first order multi-variable partial differential equation for the Helmholtz free energy F. A solution can be constructed for F through iteration, first order by order in λ and then order by order in η at a given order in λ. After some calculation, we find F to $\mathcal{O}(\lambda^2)$, and this is given by,

$$F = m_0 \left[\left(1 - \frac{d-3}{d-2}\,\eta \right) + \left(\frac{d-3}{d-2}\,\eta - \frac{1}{2}\,\eta^2 \right)\lambda + \left(-\frac{d-3}{d-2}\,\eta + \frac{1}{2}\,\eta^2 - \frac{1}{2}\frac{d-3}{d-2}\,\eta^3 \right)\lambda^2 \right]. \tag{B.7}$$

The entropy S, the electrostatic potential Φ, and the tension $\hat{\tau}$ are computed using the thermodynamic relations in $(B.2)$–$(B.4)$, and their results agree with the ones stated in section 3.2. Using a Legendre transformation, the Gibbs free energy is $G = F - \Phi Q$ and we find agreement with (3.27).

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